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RETURN CURRENT INDUCED BY A RELATIVISTIC BEAM PROPAGATING IN A MAGNETIZED PLASMA

by

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Abstract

The propagation of a high current relativistic beam in a cold magnetized plasma is investigated using a model developed by Hammer and Rostoker for the unmagnetized case. In this model, the beam electrons are assumed to be undeflected from their zero order orbits and the fields associated with the beam are switched on at time $t=0$. The return current induced in the plasma is calculated as a function of beam and plasma parameters. It is demonstrated that the return current does not extend indefinitely but dies away inversely as the distance from the head of the beam with a characteristic length $L_n = v_o \tau a^2/\lambda_E{}^2$, where v_o and a are the beam velocity and radius, τ is a phenomenological momentum relaxation time for the plasma electrons, $\lambda_E = c/\omega_p$ and ω_p is the plasma frequency of the plasma electrons. When the beam is injected either parallel or perpendicular to a static magnetic field B_o, it is found to be magnetically neutralized by a return current over a length of order L_n if $a^2/\left[\lambda_E{}^2(1 + \Omega^2/\omega_p{}^2)\right] >> 1$, where $\Omega = eB_o/m_oc$.

I. INTRODUCTION

The last few years have seen a tremendous increase in interest in relativistic electron beams with several experimental groups propagating high current relativistic beams[1-5]. However, early workers in this field predicted that electrostatically neutralized beams with uniform current density would not propagate with total current greater than 17,000 $\beta\gamma$ amperes,[6,7] where β is the particle stream velocity divided by the velocity of light, and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$. This current limit is due to the self-magnetic field of the beam which for large current becomes strong enough to turn the beam particles around. One way of overcoming this limit is to inject the beam into a plasma. Under certain conditions a return current will be induced to flow in the plasma within the beam. The magnetic field is then reduced and in some cases almost completely canceled. Hammer and Rostoker,[8] and Cox and Bennett[9] have recently developed models for calculating the properties of the induced plasma current. In this paper we will extend the model developed by Hammer and Rostoker[8] to the case of injection of a magnetized plasma.

In this model, a cylindrical relativistic beam of uniform density is "switched on" at t=0 in the presence of a fully ionized plasma. The fields generated by this beam are computed as if the test particles composing the beam are unperturbed from their zero order motion. The response of the plasma to the beam is calculated by means of a plasma conductivity tensor which is developed in the conventional manner from Maxwell's equations and the appropriate dynamical equations for the plasma. The current induced in the plasma by the time changing fields associated with the head of the beam is computed as a function of the beam and plasma parameters. It is possible to compute the effect on the beam particles of the fields generated by the beam and thereby obtain a better approximation to the behavior of the beam and plasma. However, in this paper we shall confine our calculations to the first step in this perturbation expansion.

In Sec. II we develop the general expression for the induced plasma current for a beam injected at an arbitrary angle into a uniformly magnetized plasma. In Sec. III we briefly discuss the zero magnetic field case treated by Hammer and Rostoker.[8] For this one finds the following picture for the plasma current. At the head of the beam (see Fig. 1) there is a region of plasma oscillations which decay away with a scale length of $O(v_o\tau)$, where v_o is the beam velocity and τ is a phenomenological momentum relaxation time for the plasma electrons. If $a^2/\lambda_E^2 >> 1$, where a is the beam radius, $\lambda_E = c/\omega_p$, the electromagnetic skin depth, and ω_p is the plasma frequency of the background plasma, then a region exists where the beam current is neutralized. The length of the neutralized region, L_n, is of $0(v_o \tau a^2/\lambda_E^2)$ and the net current is approximately λ_E/a times the beam current. The length over which the beam is current neutralized, can be calculated by several elementary methods but the simplest picture is to consider the time required for the magnetic field from the plasma current to diffuse away.[10] The diffusion of the magnetic field is governed by

$$\frac{4\pi\sigma}{c^2}\frac{\partial B}{\partial t} = \nabla^2 B \quad , \tag{1.1}$$

where the conductivity $\sigma = \omega_p^2 \tau/4\pi$. If we approximate $\nabla^2 B$ by B/a^2, we see that B will decay exponentially with a time constant T given by

$$T = \tau a^2 \omega_p^2/c^2 \quad . \tag{1.2}$$

This time then gives the characteristic diffusion length, $L_n = v_o T = v_o \tau a^2/\lambda_E^2$. In the detailed calculations given in Sec. III it turns out that although the characteristic time is given by (1.2) the decay is not exponential but algebraic. Thus, once the plasma current and magnetic field diffuse away, the beam will no longer be magnetically neutralized and the beam magnetic field would exert a self pinching force. One would not expect to find beams with total length much greater than L_n if the beam current is greater than the Alfven-Lawson limit.[6,7]

In Sec. IV we consider injection of the beam parallel to the magnetic field for two sets of beam-plasma parameters; (a) $\zeta \sim 1$, $\gamma >> 1$, and (b) $\zeta >> 1$, $\gamma \sim 1$, where $\zeta = \Omega_{\parallel}/\omega_p$, and $\Omega_{\parallel} = eB_{\parallel}/m_o c$. In case (a) the magnetic field has little effect and the properties of the return current are qualitatively the same as for the zero magnetic field case discussed in Sec. III. However, in case (b) we show that the magnetic field does play an important role and if we take the limit $\zeta \to \infty$ we can immediately show that no return current will flow. Physically, the lack of return current comes about for the following reason. Unlike the case where the magnetic field is zero or weak, i.e. $0 < \zeta \lesssim 1$, there are no nonadiabatic processes available (except weak collisions) to the plasma which will allow the excess electrons to escape in the radial direction and neutralize the beam charge density. The only direction available for the excess electrons to escape is along the field lines and out the head of the beam. These electrons then pile up in front of the beam creating a large potential which prevents the plasma electrons from being accelerated backwards to provide a return current for the beam. We show that the infinite magnetic field case is a true limiting case and that the appropriate criterion for current neutralization is now $a^2/\lambda_E^2(1 + \xi^2) >> 1$. If this criterion is met, a return current will be induced and the length of the current neutralization will be the same as before.

In Sec. V we consider injection of the beam perpendicular to the magnetic field for the case where $\xi \sim 1$ and $\gamma >> 1$, where $\xi = \Omega_\perp/\omega_p$, and $\Omega_\perp = eB_\perp/m_o c$. If the beam is to be current neutralized the return current must be induced to flow across the magnetic field. The mechanism which allows the electrons to flow across the magnetic field lines is provided by a space charge electric field which builds up when the beam expels excess plasma electrons in the process of becoming charge neutral. This electric field has just the proper character to cancel the magnetic force on the return current electrons.[11] The electrons then move in a force free region and the return current flows until it diffuses away by collisions. The criterion for neutralization is given by $a^2/\lambda_E^2(1 + \xi^2) >> 1$. The current neutralization length is the same as before. In summary, table I gives the conditions for beam current neutralization for the three cases discussed, i.e. injection of the beam into an unmagnetized plasma, injection parallel to a magnetic field, and injection perpendicular to a magnetic field.

In order to obtain some feeling for the numeric values of the parameters in table I we will consider conditions which are representative of the relativistic electron beam experiments at Cornell University.[1] Typically, we have beams with current of the order of 100-kA and radius of 5 cm. The beam is injected into air at a pressure of about 1 torr, and has a pulse length of 50-nsec. The background plasma density quickly rises to 10^{13} cm^{-3} or above and the beams are observed to be current neutralized. Under these conditions a/λ_E is approxiamtely 30 and we expect the beam to be current neutralized. The scale length for the damping of the plasma oscillations, $v_o\tau$, is 100 cm or less and is small compared to the total beam length (15m). The neutralization length L_n calculated from classical two particle collisions is about 10^3m, which means that the beam will be neutralized over its entire length. It is possible for the effective collision frequency $1/\tau$ to be considerably greater than the classical value because of turbulence generated in the background plasma by collective interaction of the beam-plasma system. In this case, L_n would be considerably reduced.

Recently, experiments that involve the injection of beams parallel to a magnetic guide field have begun.[12] The effect of the magnetic field on the current neutralization would be noticeable if $a/\lambda_E\zeta \sim 1$; with $a/\lambda_E \sim 30$, and $\zeta \sim 30$, the magnetic field must be about 300-kg. To keep $a/\lambda_E \sim 10$ and still see some effect from a 10-kg guide field one needs a plasma density of 10^{11} cm^{-3} and a beam radius of 17 cm. These conditions just represent the threshold of the effect and guide fields 5 to 10 times the above strength would be needed before beam neutralization was significantly affected. However, even though the parallel guide field may be strong enough to prevent current neutralization the propagation of the beam may not be affected since the guide field will most likely now be much stronger than the self magnetic field of the beam. If this is true the guide field will prevent the beam particles from being turned around by the self field and the beam will propagate.[12]

At the Astron facility of the Lawrence Radiation Laboratory, relativistic electron beams are used to create an electron E-layer in a magnetic mirror configuration.[13] To a first approximation we may treat the injection into this machine as injection of an electron beam across a magnetic field. For purposes of discussion, we will consider the injection of an Astron beam under the conditions of steady state E-layer production. That is, an E-layer has been formed by several earlier pulses and the background neutral Hydrogen (say at a pressure of 10^{-3} torr) is now fully ionized. The electron beam, with energy around 4-MeV, radius of 3 cm, current of 1-kA, and pulse length of 300-nsec is then injected into the magnetic mirror; a/λ_E is again about 30 and $\xi = \Omega_{\perp}/\omega_p \approx 10^{-2}$. Thus we see that the magnetic field has little effect on the beam which will be fully current neutralized upon injection. Once the head of the beam has wound its way into the E-layer and is reflected at the far mirror, it will lose its identity and will no longer induce a return current. The neutralization length is 300 m and therefore it will take about 10^{-6} sec before the beam will contribute its full magnetic field to the E-layer.

II. MATHEMATICAL FORMULATION

In this section we will outline the mathematical formalism used in calculating the response of the plasma to the injected beam of electrons. The beam is injected into a cold magnetized plasma assumed to be homogeneous and infinite in extent. The injection process is treated as an initial value problem where the beam is "switched on" at time t=0 in a quiescent plasma. It becomes convenient to do the calculation in the beam rest frame because we are solving an initial value problem. It is in this frame of reference that the initial conditions have their simplest form, i.e., the plasma is assumed to be unperturbed at t=0 when the beam is switched on and the only fields present are the electrostatic fields associated with the unneutralized charge density of the beam.

We perform a Fourier transform on Maxwell's equations in space and a Laplace transform in time and get

$$i\underset{\sim}{k} \times \underset{\sim}{E}(s,\underset{\sim}{k}) = -\frac{s}{c}\,\underset{\sim}{B}(s,\underset{\sim}{k}), \tag{2.1}$$

and

$$i\underset{\sim}{k} \times \underset{\sim}{B}(s,\underset{\sim}{k}) = \frac{4\pi}{c}\,\underset{\sim}{j}(s,\underset{\sim}{k}) + \frac{s}{c}\,\underset{\sim}{E}(s,\underset{\sim}{k}) - \frac{1}{c}\,\underset{\sim}{E}(t=0,\underset{\sim}{k}), \tag{2.2}$$

where

$$\underset{\sim}{E}(s,\underset{\sim}{k}) = \int_0^{\infty} ds \int_{-\infty}^{+\infty} d^3x\; \underset{\sim}{E}(x,\underset{\sim}{k}) \exp\,(-st-i\underset{\sim}{k}\cdot\underset{\sim}{x}), \tag{2.3}$$

etc. Combining Eqs. (2.1) and (2.2) we obtain

$$-\underset{\sim}{k} \times \underset{\sim}{k} \times \underset{\sim}{E}(s,\underset{\sim}{k}) + \frac{s^2}{c^2}\underset{\sim}{E}(s,\underset{\sim}{k}) + \frac{4\pi s}{c^2}\underline{\underline{\sigma}}(s,\underset{\sim}{k})\cdot\underset{\sim}{E}(s,\underset{\sim}{k}) =$$

$$\frac{s}{c^2}\underset{\sim}{E}(t=0,\underset{\sim}{k}) \quad . \tag{2.4}$$

We have written $\underset{\sim}{j}(s,\underset{\sim}{k})$ as a tensor product between a conductivity tensor $\underline{\underline{\sigma}}(s,\underset{\sim}{k})$ and the electric field $\underset{\sim}{E}(s,\underset{\sim}{k})$.

If we assume that the thermal velocity of the plasma electrons can be neglected, in comparison with the beam velocity, then the equation for the electron fluid momentum is

$$\frac{d}{dt}\underset{\sim}{p}(\underset{\sim}{x},t) = -e\left[\underset{\sim}{E}(\underset{\sim}{x},t) + \frac{1}{c}\underset{\sim}{v}(\underset{\sim}{x},t) \times \underset{\sim}{B}(\underset{\sim}{x},t)\right] - \frac{1}{\tau}\underset{\sim}{p}(\underset{\sim}{x},t) , \tag{2.5}$$

where $\underset{\sim}{p}(\underset{\sim}{x},t) = m_o\gamma(\underset{\sim}{x},t)\,\underset{\sim}{v}(\underset{\sim}{x},t)$, $\gamma(\underset{\sim}{x},t) = \{1-v(\underset{\sim}{x},t)^2/c^2\}^{-\frac{1}{2}}$, and τ is a phenomenological electron momentum relaxation time. Ion motion is neglected since the ions, due to their large mass, respond much more slowly than the electrons. In the beam rest frame the plasma is streaming in the negative z direction with speed v_o. Perturbations are assumed to be of order $N_b/N_o << 1$, where N_b and N_o are the beam and background plasma densities. This restriction ensures that the perturbed velocity of the plasma electrons remains small with respect to the beam velocity v_o. Linearizing

Eq. (2.5) about v_o and taking the Fourier-Laplace transform of this equation (2.5) we get

$$\gamma_o(s-ik_z v_o)\left[\underset{\sim}{v}_\perp^{(1)}(s,\underset{\sim}{k}) + \gamma_o^2 \underset{\sim}{v}_\parallel^{(1)}(s,\underset{\sim}{k})\right]$$

$$= -\frac{e}{m}\left[\underset{\sim}{E}^{(1)}(s,\underset{\sim}{k}) - \frac{v_o}{c}\,\hat{a}_z \times \underset{\sim}{B}^{(1)}(s,\underset{\sim}{k}) + \frac{1}{c}\,\underset{\sim}{v}^{(1)}(s,\underset{\sim}{k}) \times \underset{\sim}{B}_o\right]$$

$$- \frac{\gamma_o}{\tau}\left[\underset{\sim}{v}_\perp^{(1)}(s,\underset{\sim}{k}) + \gamma_o^2\, \underset{\sim}{v}_\parallel^{(1)}(s,\underset{\sim}{k})\right] \quad . \qquad (2.6)$$

Superscripts refer to perturbed quantities; $\underset{\sim}{v}_\perp^{(1)}(s,\underset{\sim}{k})$ is the perturbed velocity normal to the z axis and $\underset{\sim}{v}_\parallel^{(1)}(s,\underset{\sim}{k})$ is the perturbed velocity parallel to the z axis, and $\gamma_o = (1-v_o^2/c^2)^{-\frac{1}{2}}$. All quantities are measured in the beam rest frame unless noted otherwise.

Equation (2.6) is easily solved for the perturbed velocity $\underset{\sim}{v}^{(1)}(s,\underset{\sim}{k})$. Using the equation of the conservation of number density, the perturbed current is then given by

$$\underset{\sim}{j}^{(1)}(s,\underset{\sim}{k}) = -eN_o \underset{\sim}{v}^{(1)}(s,\underset{\sim}{k}) - \frac{ieN_o \underset{\sim}{k}\cdot\underset{\sim}{v}^{(1)}(s,\underset{\sim}{k})}{s-ik_z v_o}\,\hat{a}_z \quad . \qquad (2.7)$$

It is now convenient to change the basis vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ to a new set $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ where

$$\hat{e}_1 = k/k,\ \hat{e}_2 = \frac{1}{kk_\perp} k \times (\hat{a}_z \times k),\ \hat{e}_3 = \frac{1}{k_\perp} \underset{\sim}{k} \times \hat{a}_z.$$

The current is now expressed as (dropping the superscripts)

$$j(s,\underset{\sim}{k}) = \frac{e^2 N_o}{\gamma_o m_o s} \underline{\underline{S}}(s,\underset{\sim}{k}) \cdot E(s,\underset{\sim}{k}) \tag{2.8}$$

where the elements of the tensor $\underline{\underline{S}}(s,\underset{\sim}{k})$ are given in Appendix A. Putting Eq. (2.8) into Eq. (2.4) we get

$$\underline{\underline{Y}}(s,k) \cdot E(s,\underset{\sim}{k}) = \frac{s}{c^2} E(t=0,\underset{\sim}{k}), \tag{2.9}$$

where the operator

$$\underline{\underline{Y}}(s,\underset{\sim}{k}) \equiv \left\{ -\underset{\sim}{k} \times \underset{\sim}{k} \times \quad + \left[s^2/c^2 + (\omega_p{}^2/c^2)\ \underline{\underline{S}}(s,\underset{\sim}{k}) \right] \right\} \quad ,$$

and $\omega_p{}^2 = 4\pi N_o e^2/\gamma_o m_o$.

The electric field in the plasma can now be expressed in terms of the field at t=0,

$$E(s,k) = \frac{s}{c^2} \underline{\underline{Y}}(s,\underset{\sim}{k})^{-1} \cdot \underset{\sim}{E}(t=0,\underset{\sim}{k}). \tag{2.10}$$

The operator $\underline{\underline{Y}}(s,\underset{\sim}{k})^{-1}$ is given in Appendix B in terms of the cofactor matrix, $\underline{\underline{R}}(s,\underset{\sim}{k})$, and the determinant of $\underline{\underline{Y}}(s,\underset{\sim}{k})$. From Eqs. (2.8) and 2.10) the current induced in the plasma can also be given in terms of $\underset{\sim}{E}(t=0,\underset{\sim}{k})$,

$$j(s,k) = \frac{1}{4\pi}\frac{\omega_p^2}{c^2}\frac{\underset{\approx}{Q}(s,k)\cdot E(t=0,k)}{|\underset{\approx}{Y}(s,k)|} \qquad (2.11)$$

where $Q(s,k) = \underset{\approx}{S}(s,k)\cdot R(s,k)$. See appendix C for the elements of $\underset{\approx}{Q}(s,k)$. From Eq. (2.11) and the equation for charge conservation we can calculate all components of the plasma current and the charge density.

Because we are interested in the return current in the laboratory frame for $t\to\infty$ we must transform Eq. (2.11) by a Lorentz transformation to the laboratory frame, and obtain the time asymptotic limit from the prescription $F(t\to\infty) = \lim_{s\to 0} sF(s)$. We expect the plasma to reach this asymptotic value in a few plasma periods ω_p^{-1}. The expression for the return current, which is the z-component of the plasma current, is then (primes denote quantities measured in the laboratory frame)

$$j_z'(t\to\infty,k) = \lim_{s\to 0} \gamma_o\left[\frac{sk_\perp}{k} j_2(s,k) - ikv_o j_1(s,k)\right] . \qquad (2.12)$$

Using Eq. (2.11) to evaluate $j_1(s,\underset{\sim}{k})$ and $j_2(s,\underset{\sim}{k})$ we have

$$j_z'(t\to\infty,k) = \lim_{s\to 0} -i\gamma_o \frac{\omega_p^2}{c^2}\frac{\rho_b(k)}{|Y(s,k)|}\left[\frac{sk_\perp}{k^2 v_o} Q_{21}(s,k) - iQ_{11}(s,k)\right], \qquad (2.13)$$

where $\rho_b(\underset{\sim}{k})$ is the Fourier transform of the charge density of the beam. For a uniform cylindrical beam of radius a, extending from z = 0 to z = -∞ the charge density is

$$\rho_b(\underset{\sim}{x}) = \begin{bmatrix} -eN_b, & r \leq a; \; -\infty < z \leq 0 \\ & \\ 0 \;, & \text{elsewhere} \end{bmatrix} \tag{2.14}$$

and the Fourier transform of the charge density is given by,

$$\rho_b(k) = -2\pi N_b ea \frac{J_1(k_\perp a)}{k_\perp} \int_{-\infty}^{0} d\tilde{z} \exp(ik_z\tilde{z}) . \tag{2.15}$$

The z-component of the current in the laboratory frame is furnished by the inverse Fourier transform

$$j_z'(t\to\infty, r, \theta, z) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} dk_z \int_0^{\infty} k_\perp dk_\perp \int_0^{2\pi} d\phi$$

$$\cdot \; j_z'(t\to\infty, k) \exp\left[ik_z z + ik_\perp r \cos(\phi-\theta)\right] . \tag{2.16}$$

It should be noted that even though this is the expression for the current in the laboratory frame, the coordinates are still measured in the beam frame. Another quantity of interest is the perturbed charge density. In the laboratory frame we have

$$\rho'(t\to\infty,\underset{\sim}{k}) = \lim_{s\to 0} \gamma_o \left[-ikj_1(s,\underset{\sim}{k}) + \frac{sv_o k_\perp}{c^2 k} j_2(s,\underset{\sim}{k}) \right] \tag{2.17}$$

$$= \lim_{s\to 0} -\gamma_o \frac{\omega_p^{\;2}}{c^2} \frac{\rho_b(\underset{\sim}{k})}{|\underline{\underline{Y}}(s,\underset{\sim}{k})|} \cdot \left[Q_{11}(s,\underset{\sim}{k}) + \frac{isv_o k_\perp}{c^2 k^2} Q_{21}(s,\underset{\sim}{k}) \right] . \tag{2.18}$$

Substituting Eqs. (2.13) and (2.15) into Eq. (2.16), we see that the expression for the plasma current has the following form

$$j_z'(t\to\infty,\underset{\sim}{x}) = -\left[ev_o N_b \gamma_o\right]\left(\frac{\omega_p^{\;2}}{\gamma_o^{\;2} v_o^{\;2}}\right) a \int_0^\infty \frac{dk_\perp}{2\pi} J_1(k_\perp a)$$

$$\cdot \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\left[ik_\perp r \cos(\phi-\theta)\right]$$

$$\cdot \int_{-\infty}^{0} d\tilde{z} \int_{-\infty}^{+\infty} dk_z\, k_z \frac{\prod_{i=1}^{\ell-3}(k_z-k_i)}{\prod_{j=1}^{\ell}(k_z-k_j)} \exp\left[ik_z(z-\tilde{z})\right] . \tag{2.19}$$

The zeros of the two polynomials depend, in general, on $k_\perp$ and ϕ as well as the plasma parameters ω_p, τ, and v_o; ℓ is the degree of the polynomial, $\ell = 5$ for $\underset{\sim}{B}_o = 0$ and $\ell = 8$ for $\underset{\sim}{B}_o \neq 0$. Hereafter, the plasma current and charge density will be normalized to the zero order beam current and charge density and all lengths will be normalized to $\lambda_E \equiv c/\omega_p$, the electromagnetic skin depth. We do the k_z integration by contour integration around the zeros of the polynomial in the denominator of Eq. (2.19). The $\tilde{z}$ integration is trivial. We then see that for $z > 0$ the plasma current depends only on the residues of the k_z poles which are in the upper half plane (UHP), and for $z < 0$ the current depends only on the residues in the lower half plane (LHP). Since we are interested in the plasma current induced to flow within the beam, i.e. for $z < 0$, we will not consider the expression for the current for $z > 0$ any further. In most cases the current ahead of the beam is $O\{\exp(-z/\lambda_E)\}$ and is approximately zero anyway. The expression for the normalized plasma current, after performing the k_z and $\tilde{z}$ integration is now

$$j_z'(t\to\infty, R, \theta, Z>0) = \frac{A}{\alpha^2} \int_0^\infty \frac{d\kappa}{2\pi} J_1(\kappa A) \int_0^{2\pi} d\phi \tag{2.20}$$

$$\cdot \exp\left[i\kappa R\cos(\phi-\theta)\right] \times \sum_{\substack{k \\ \text{LHP}}} \frac{\prod_{i=1}^{\ell-3} (\chi_k - \chi_i)}{\prod_{\substack{j=1 \\ j\neq k}}^{\ell} (\chi_k - \chi_j)} \exp(-i\chi_k Z)$$

where $\alpha = \beta_o\gamma_o$, $\kappa = k_\perp\lambda_E$, $\chi = k_z\lambda_E$, $A = a/\lambda_E$, $R = r/\lambda_E$, and $Z = -z/\lambda_E$, where for convenience Z is the distance measured from the head of the beam. The sum is over the residues in the LHP.

From the general structure of Eq. (2.20) can we determine if a "return current" is induced to flow in the plasma? By a "return current" we mean a quasi-static current induced within the beam that exist for a time long compared to the characteristic decay time of the plasma oscillations at the head of the beam. Since the decay time of the oscillations is τ then the plasma oscillations exist over a region at the head of the beam $Z = v_o\tau/\lambda_E$. In the beam coordinates, the return current is quasi-steady and the decay length is long compared to the decay length for the plasma oscillations at the head of the beam. Inspection of the roots of the polynomial shows that the damping associated with the oscillating terms is always as great or greater than $\nu/2$, $\nu \equiv \lambda_E/v_o\tau$. Thus the plasma oscillations terms are always $0[\exp(-Z\nu/2)]$. Therefore, we conclude that any of the poles with a damping greater than $\nu/2$ cannot contribute to the return current. If there are no poles with a damping less than $\nu/2$, then no return current will be induced. Only poles with a damping less than $\nu/2$ can give rise to a return current. To determine this current we only need to calculate the residues from these poles. Detailed examination of these roots will be undertaken for the three special cases discussed in the following sections.

III. ZERO MAGNETIC FIELD

In order to understand in what sense the return current can be considered a quasi-static current we will first treat the case of zero magnetic field, i.e. set $\underset{\sim}{B}_0 = 0$, and obtain the case treated by Hammer and Rostoker.[8] Equation (2.19) then becomes

$$j_z'(R,Z>0) = \frac{A}{\alpha^2} \int_0^\infty \frac{d\kappa}{2\pi} J_1(\kappa A)\, J_0(\kappa R) \int_{-\infty}^{0} d\tilde{Z} \int_{-\infty}^{+\infty} d\chi$$

$$\cdot \exp\left[-i(Z+\tilde{Z})\chi\right] \times \frac{\chi\left[\chi(\chi+i\nu)+1\right]}{\left[(\chi^2+\kappa^2)(\chi+i\nu)+\chi\right]\left[\chi(\chi+i\nu)-1/\alpha^2\right]} . \tag{3.1}$$

The zeros of the denominator of the integrand in Eq. (3.1) are approximately given by

$$\chi_1 = +i(\kappa^2+1)^{\frac{1}{2}} - i\tfrac{1}{2}\nu/(\kappa^2+1)$$

$$\chi_2 = -i(\kappa^2+1)^{\frac{1}{2}} - i\tfrac{1}{2}\nu/(\kappa^2+1)$$

$$\chi_3 = +1/\alpha - i\tfrac{1}{2}\nu$$

$$\chi_4 = -1/\alpha - i\tfrac{1}{2}\nu$$

$$\chi_5 = -i\nu\kappa^2/(\kappa^2+1)$$

We notice that only χ_5 will contribute to the return current since its damping is less than $\nu/2$ as discussed in the previous section. χ_3 and χ_4 give rise to exponentially damped oscillations which when transferred to the laboratory frame are exactly at the plasma frequency; χ_2 gives rise to a nonoscillatory term which decays with a very rapidly exponential character. χ_1 is in the UHP and so does not contribute. In Hammer and Rostoker's calculation χ_5 is put equal to zero which gives a return current that does not decay away. On calculating the residue for χ_5, the expression for the return current is

$$j_z'(R,Z<0) = -A \int_0^\infty d\kappa \, \frac{J_0(\kappa R) J_1(\kappa A)}{(\kappa^2 + 1)} \exp\left[-\nu Z \kappa^2/(\kappa^2+1)\right] \quad . \tag{3.2}$$

We approximate the integral in the following way. If one assumes there is a range of Z where the exponential in Eq. (3.2) is approximately equal to unity, the integral can then be done and the current is

$$j_z'(R,Z_1<Z<Z_2) = \begin{cases} -1 + AI_0(R)K_1(A), & R<A \\ \\ -AI_1(A)K_0(R), & R>A \end{cases} \quad , \tag{3.3}$$

where the I's and K's are modified Bessel functions. The ratio of the net current within the beam to the beam current is

$$I_{net}/I_{beam} = \frac{1}{\pi A^2}\int_0^A dR\ 2\pi RAI_o(R)K_1(A) = 2I_1(A)K_1(A) \quad . \tag{3.4}$$

From Eq. (3.4) we conclude that for effective neutralization, i.e. $I_{net}/I_{beam} << 1$, we must have

$$A = a/\lambda_E >> 1 \quad . \tag{3.5}$$

In this limit

$$I_{net}/I_{beam} \approx \frac{1}{A} \tag{3.6}$$

This is the result obtained by Hammer and Rostoker.

Returning to Eq. (3.2), we are interested in the Z dependence of the current for $Z>1/\nu$, i.e. the region after the plasma oscillations have decayed away. Thus, the only important part of the integration lies in the range $0<\kappa<1$ since the integral in the range $1<\kappa<\infty$ is $0(\exp -Z\nu/2)$. For effective neutralization $A>>1$, which makes $J_1(\kappa A)$ oscillate quite rapidly for $\kappa>1/A$, and here little error will be introduced if we neglect κ^2 with respect to unity in Eq. (3.2). We let the range of the integration go to ∞ because $J_1(\kappa A)$ oscillates quite rapidly and the exponential will be quite strongly damped for $Z>1/\nu$. The return current is then approximately given by

$$j_z'(R, Z > 1/\nu) \approx -A \int_0^\infty d\kappa \, J_o(\kappa R) J_1(\kappa A) \exp(-\nu Z \kappa^2) . \quad (3.7)$$

This integral still cannot be done in closed form except for the special case of R = 0. From Eq. (3.3), however, we expect the current to be approximately uniform across the cross section of the beam since for A>>1, the modified Bessel functions are exponentially small except at R = A. For R = 0, Eq. (3.7) becomes

$$j_z'(R, Z > 1/\nu) \approx -2\exp(-A^2/8\nu Z) \sinh(A^2/8\nu Z)$$

$$= -\left[1 - \exp(-A^2/4\nu Z)\right] . \quad (3.8)$$

We now see that the current decays away quite slowly with a scale length $L_n \sim 0(A^2/\nu)$. Note also that for large Z the current decays as Z^{-1}. We, therefore, conclude that there is a region where the return current is in a quasi-static state. The scale length of this region is much longer than the scale length of the damping of the plasma oscillations, and provides a region of current neutralization which is necessary for ultrahigh current beams to propagate.

IV. INJECTION PARALLEL TO MAGNETIC FIELD

In this section we will consider the injection of the beam parallel to the static magnetic field. Equation (2.19) then becomes

$$j_z'(R,Z) = \frac{A}{\alpha^2} \int_0^\infty \frac{d\kappa}{2\pi} J_o(\kappa R) J_1(\kappa A) \int_{-\infty}^{0} d\tilde{Z} \int_{-\infty}^{+\infty} d\chi \exp\left[-i(Z+Z)\chi\right]$$

$$\cdot \left\{ \chi\left[(\chi^2+\kappa^2)(\chi+i\nu)+\chi\right]\left[\chi(\chi+i\nu)+1\right] - \frac{\zeta^2}{\alpha^2}\chi^2(\chi^2+\kappa^2)\right\}$$

$$\cdot \left\{\left[(\chi^2+\kappa^2)(\chi+i\nu)+\chi\right]^2\left[\chi(\chi+i\nu)-1/\alpha^2\right] - \frac{\zeta^2}{\alpha^2}\chi(\chi^2+\kappa^2)\right.$$

$$\left.\cdot\left[(\chi^2+\kappa^2)(\chi+i\nu)-\chi/\alpha^2\right]\right\}^{-1} , \qquad (4.1)$$

where $\zeta = \Omega_{\parallel}/\omega_p$, and $\Omega_{\parallel} = eB_{oz}/m_oc$. The polynomial in the denominator of the integrand is of the eighth degree and its zeros, in general, are not easily found. However, we will consider two cases where they can be calculated using an expansion in a small parameter, ε; (a) $\zeta \sim 1$, $\alpha^{-1} \sim \varepsilon$, and $\nu \sim \varepsilon^2$ and (b) for $\zeta^{-1} \sim \varepsilon$, $\alpha \sim 1$, and $\nu \sim \varepsilon^2$. We will first consider case (a). Keeping only lowest order real and imaginary terms in each zero we have, for the first case,

$$\chi_1 = \chi_2 = +i(\kappa^2+1)^{1/2}$$

$$\chi_3 = \chi_4 = -i(\kappa^2+1)^{1/2}$$

$$\chi_5 = +\frac{1}{\alpha}\left[1+\zeta^2 f^2(\kappa^2)\right] - i\nu\left[\tfrac{1}{2}+\zeta^2 f^3(\kappa^2)\right]$$

$$\chi_6 = -\frac{1}{\alpha}\left[1 + \zeta^2 f^2(\kappa^2)\right] - i\nu\left[\tfrac{1}{2} + \zeta^2 f^3(\kappa^2)\right]$$

$$\chi_7 = -i\nu \frac{f(\kappa^2)}{1 + \zeta^2 f^2(\kappa^2)}\left[1 + \tfrac{1}{2}\zeta^2 f(\kappa^2)\left\{1 + \left[1-2(\kappa^2-1)/\zeta^2\kappa^2\right]^{\frac{1}{2}}\right\}\right]$$

$$\chi_8 = -i\nu \frac{f(\kappa^2)}{1 + \zeta^2 f^2(\kappa^2)}\left[1 + \tfrac{1}{2}\zeta^2 f(\kappa^2)\left\{1 - \left[1-2(\kappa^2-1)/\zeta^2\kappa^2\right]^{\frac{1}{2}}\right\}\right],$$

where $f(\kappa^2) = \kappa^2/(\kappa^2 + 1)$. We now see that only χ_7 and χ_8 are weakly damped and by virtue of the argument presented in Section II, they only contribute to the return current. Calculating the residues from these poles, the expression for the return current is

$$j_z'(R,Z) = -A\int_0^\infty d\kappa \; \frac{J_0(\kappa R)J_1(\kappa A)}{(\kappa^2 + 1)\left[1 + \zeta^2 f^2(\kappa^2)\right]} \tag{4.2}$$

$$\cdot \left[\frac{(\chi_7 - \chi_9)\exp(-i\chi_7 Z) - (\chi_8 - \chi_9)\exp(-i\chi_8 Z)}{(\chi_7 - \chi_8)}\right],$$

where $\chi_9 = -i\nu f(\kappa^2)$. If we put $\zeta = 0$ in Eq. (4.2), it will reduce to the expression for no magnetic field, Eq. (3.22). Again we will assume there is a range of Z where the exponentials are approximately equal to unity, i.e. we neglect altogether the damping introduced by χ_7 and χ_8. In the following paragraph we will explicitly determine the effect of this damping on the decay of the return current. Equation (4.2) then becomes

$$j_z'(R,Z) = -A \int_0^\infty d\kappa \frac{J_0(\kappa R) J_1(\kappa A)}{(\kappa^2 + 1)\left[1 + \zeta^2 f^2(\kappa^2)\right]}$$

$$= \begin{cases} -1 + \dfrac{A}{2(1 + i\zeta)^{\frac{1}{2}}} I_0\left[R/(1 + i\zeta)^{\frac{1}{2}}\right] K_1\left[A/(1 + i\zeta)^{\frac{1}{2}}\right] + \text{c.c.} & R<A \\ -\dfrac{A}{2(1 + i\zeta)^{\frac{1}{2}}} I_1\left[A/(1 + i\zeta)^{\frac{1}{2}}\right] K_0\left[R/(1 + i\zeta)^{\frac{1}{2}}\right] + \text{c.c.} & R>A \end{cases} \quad (4.3)$$

The ratio of the net current within the beam to the beam current is

$$\frac{I_{net}}{I_{beam}} = \frac{1}{\pi A^2} \int_0^A dR \frac{2\pi RA}{2(1 + i\zeta)^{\frac{1}{2}}} I_0\left[R/(1 + i\zeta)^{\frac{1}{2}}\right] \cdot K_1\left[A/(1 + i\zeta)^{\frac{1}{2}}\right]$$

$$+ \text{c.c} = I_1\left[A/(1 + i\zeta)^{\frac{1}{2}}\right] K_1\left[A/(1 + i\zeta)^{\frac{1}{2}}\right] + \text{c.c.} \quad . \quad (4.4)$$

For $A >> 1$, the condition for current neutralization, we have

$$I_{net}/I_{beam} \approx \frac{1}{A}\left[\frac{(1 + i\zeta)^{\frac{1}{2}} + (1 - i\zeta)^{\frac{1}{2}}}{2}\right] \quad (4.5)$$

which is approximately the result we obtained without a magnetic field.

Because the terms in the square brackets in Eq. (4.2) are complicated functions of κ^2, the κ integration cannot be done exactly for arbitrary Z. However, what we are most interested in is the effect of the magnetic field on the return current, and how this return current is different from that induced with no magnetic field present. We have just seen that in the current neutralized region, for case (a), the magnetic field has only a weak effect on the percent neutralization. To estimate the contribution of the magnetic field to χ_7 and χ_8, for current neutralized beams, we note that the most important values of κ in the κ integration are in the range $0 < \kappa < 1/A$. In this range $f(\kappa^2)$ is always less than $1/A^2$. Therefore, for a current neutralized beam $1/A^2 \ll 1$, $f(\kappa^2) \ll 1$, and the terms in χ_7 and χ_8 proportional to the magnetic field, ζ, can be neglected compared to unity. Equation (4.2) now reduces to

$$j_z'(R,Z) = -A \int_0^\infty d\kappa \, \frac{J_0(\kappa R) J_1(\kappa A)}{(\kappa^2 + 1)\left[1 + \zeta^2 f^2(\kappa^2)\right]} \cdot \exp\left[-\nu Z \kappa^2/(\kappa^2 + 1)\right] \quad . \tag{4.6}$$

Using the same arguments as in the case for injection into a unmagnetized plasma, we conclude that the current decays away with the same scale length, $L_n \sim A^2/\nu$, and that for large Z the return current decays at Z^{-1}. Thus, under this first set of conditions we see that there is essentially no difference in return current for a magnetized or unmagnetized plasma.

The second ordering we wish to consider is the large magnetic field case (b), i.e. $\zeta^{-1} \sim \varepsilon$. Unfortunately, there is no uniformly valid perturbation expansion for all κ with this ordering. At values of κ approximately given by $\kappa \approx \alpha/\zeta$, $1/\alpha$, a set of poles, which are a conjugate pair in the $X = i\chi$ plane, converge on the real axis (the imaginary axis in the χ plane). For κ in the region near these two values the character of the two poles changes dramatically and the perturbation expansion fails. One can, of course, calculate the values of these poles at the singular points, and the positions of all the poles in the χ plane are known for all κ. One lacks, however, a good method for joining the various regions of κ space across the singular points. Because of this difficulty only approximate expressions for the return current are available.

Since the general expression for the return current is quite intractable, we will first look at a limiting case, $\zeta \to \infty$, where the complexity is considerably reduced. In this limit Eq. (4.1) becomes

$$j_z'(R,Z) = \frac{A}{\alpha^2}\int_0^\infty \frac{d\kappa}{2\pi} J_o(\kappa R) J_1(\kappa A) \int_{-\infty}^{0} d\tilde{Z} \int_{-\infty}^{+\infty} d\chi$$

$$\cdot \exp\left[-i(Z + Z)\chi\right] \frac{\chi}{\left[(\chi^2 + \kappa^2)(\chi + i\nu) - \chi/\alpha^2\right]} \qquad (4.7)$$

The polynomial in the denominator of the integrand in Eq. (4.7) is a cubic, and the zeros can be found from exact, but complicated, expressions. One then finds that there are no poles in Eq. (4.7) whose damping is less than $-i\nu/2$, and thus there will be no return current in this limit. Physically, the lack of return current comes about for the following reason. Unlike the case where the magnetic field is zero or weak, i.e. $0 \leq \zeta \leq 1$, there are no nonadiabatic processes available (except weak collisions) to the plasma which will allow the excess plasma electrons to escape in the radial direction and neutralize the beam charge density. The only directions available for the excess electrons to escape is along the field lines and out the head of the beam. These electrons then pile up in front of the beam creating a large potential which prevents the plasma electrons ahead of the beam from being accelerated to provide return current for the beam. To see that the infinite magnetic field case is a true limiting case and not a singular result, we will now return to the discussion of the large, but finite, magnetic field ordering.

Inspection of the zeros of the polynomial in the integrand of Eq. (4.1) for the large magnetic field ordering (See Appendix D for a detailed description of the zeros) shows that there are six zeros in the LHP, but only two have a range of K where they are weakly damped. These two zeros are located at the origin for $\kappa = 0$, and move down the imaginary axis as κ increases. At first they move together and for $\kappa < \alpha/\zeta$ they are equal to

$$\chi_1 \approx \chi_2 = -i\nu\kappa^2 \qquad . \tag{4.8}$$

At the same time there is a zero in the numerator which exactly cancels one of these poles. As κ increases, one of the poles quickly reaches its asymptotic value and it can be expressed approximately as $\chi_1 \approx -i\nu\kappa^2/(\zeta^2\kappa^2 + 1)$ for all κ. The other pole moves down the imaginary axis monotonically to $-i\infty$. It is always approximately canceled by a zero in the numerator. If $A > \zeta > 1$ then the major contribution to the κ integration comes in the region where χ_1 and χ_2 are given by Eq. (4.8) and the return current should be qualitatively the same as in the weak magnetic field case. For $\kappa \leq 1/A$ the residue is approximately equal to 1 and the approximate expression for the return current is

$$j_z'(R,Z) \approx -A \int_0^{1/\zeta} d\kappa \, J_0(\kappa R) J_1(\kappa A) \exp\left[-\nu Z\kappa^2/(\zeta^2\kappa^2 + 1)\right] + 0(1/\zeta^2) \qquad . \tag{4.9}$$

For R = 0 and A > ζ > 1, Eq. (4.9) is approximately

$$j_z'(R = 0, Z) \approx -\frac{A^2}{2} \int_0^{2/A} \kappa d\kappa \exp(-\nu Z\kappa^2)$$

$$= -\frac{A^2}{4\nu Z}\left[1 - \exp(-4\nu Z/A^2)\right] \tag{4.10}$$

The return current has the same characteristic decay length as before, $L_n \sim A^2/\nu$, and the same Z^{-1} dependence for large Z. The return current is 0(1). For ζ > A > 1 we have

$$j_z'(R = 0, Z) \approx -\frac{A^2}{2} \int_0^{1/\zeta} \kappa d\kappa \exp\left[-Z\kappa^2/(\zeta^2\kappa^2 + 1)\right]$$

$$= -\frac{A^2}{4\zeta^2} \int_0^1 dy \exp\left[-\frac{Z\nu y}{\zeta^2(1 + y)}\right] \quad . \tag{4.11}$$

This integral can be put into the form of an incomplete Gamma function and for large Z the current is

$$j_z'(R = 0, Z\to\infty) \sim -\frac{A^2}{4\nu Z}\left[1 - 4\exp(-\nu Z/2\zeta^2)\right] \quad . \tag{4.12}$$

We again see that the current has the characteristic decay length, $L_n \sim A^2/\nu$, and goes as Z^{-1} for large Z. However, the return current is no longer 0(1) but it is $0(A^2/\zeta^2)$. Thus we have only partial current neutralization of the beam, and this neutralization goes to zero as $\zeta\to\infty$. We can now conclude that under the condition,

$A^2/\zeta^2 >> 1$, current neutralization will take place. More correctly, we should have $A^2/(\zeta^2 + 1) >> 1$.

V. INJECTION PERPENDICULAR TO MAGNETIC FIELD

In this section we will discuss the injection of the beam perpendicular to the static magnetic field. As in the preceding sections we will assume that the unperturbed orbits of the beam electrons are straight lines. This means we have assumed the gyro radius of the beam electrons to be large compared to all other transverse lengths, the beam radius, gyro radius of the background plasma, etc. With this restriction in mind, Eq. (2.19) becomes

$$j_z'(R,\theta,Z) = \frac{A}{\alpha^2} \int_0^\infty \frac{d\kappa}{2\pi} J_1(\kappa A) \int_0^{2\pi} \frac{d\emptyset}{2\pi} \exp\left[i\kappa R \cos(\emptyset-\theta)\right]$$

$$\cdot \int_{-\infty}^{0} d\tilde{Z} \int_{-\infty}^{+\infty} d\chi \exp\left[-i(Z+\tilde{Z})\chi\right] \tag{5.1}$$

$$\cdot\left\{\left[\chi(\chi^2+\kappa^2)(\chi+i\nu)+\chi\right]\left[\chi(\chi+i\nu)+1+i\xi\kappa\,\beta_o^{-1}\sin\emptyset\right]\right\}$$

$$\cdot\left\{\left[(\chi^2+\kappa^2)(\chi+i\nu)+\chi\right]^2\left[\chi(\chi+i\nu)-1/\alpha^2\right]-\frac{\xi^2}{\alpha^2}\chi(\chi^2+\kappa^2)\right.$$

$$\left.\cdot\left[(\chi^2+\kappa^2)(\chi+i\nu)+\chi\right]+\frac{\xi^2}{\alpha^4}(\chi^2+\kappa^2)\kappa^2\cos^2\emptyset\right\}^{-1},$$

where $\xi = \Omega_\perp/\omega_p$, and $\Omega_\perp = eB_{ox}/m_o\gamma_o c$.

Integration of Eq. (5.1) for arbitrary plasma conditions is now more difficult, in general, than the parallel magnetic field case. This is because of the added difficulty of the $\cos^2\emptyset$ term in the denominator of the integrand.

However, if we consider the case of a highly relativistic beam, i.e. $\alpha >> 1$, then we can neglect the $\cos^2\phi$ term since it is $0(1/\alpha^4)$. We will also order $\xi \sim 1$ and $\nu \sim \varepsilon^2$. Under these restrictions Eq. (5.1) reduces to

$$j_z'(R,\theta,Z) = \frac{A}{\alpha^2} \int_0^\infty \frac{d\kappa}{2\pi} J_1(\kappa A) \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\left[i\kappa R \cos(\phi - \theta)\right]$$

$$\cdot \int_{-\infty}^{0} dZ \int_{-\infty}^{+\infty} d\chi \exp\left[-i(Z + Z)\chi\right] \qquad (5.2)$$

$$\cdot \frac{\chi\left[\chi(\chi + i\nu) + 1 + i\xi\kappa\beta_o^{-1}\sin\phi\right]}{\left[(\chi^2 + \kappa^2)(\chi + i\nu)+\chi\right]\left[\chi(\chi + i\nu)-1/\alpha^2\right]-(\xi^2/\alpha^2)\chi(\chi^2 + \kappa^2)} .$$

The zeros in the denominator of Eq. (5.2) can now be calculated using a perturbation expansion. Keeping only the lowest order terms in the real and imaginary parts of each zero, they are

$$\chi_1 = +i(\kappa^2 + 1)^{\frac{1}{2}}$$

$$\chi_2 = -i(\kappa^2 + 1)^{\frac{1}{2}}$$

$$\chi_3 = + \frac{1}{\alpha}\left[1 + \xi^2 f(\kappa^2)\right]^{\frac{1}{2}} -i\nu/2$$

$$\chi_4 = - \frac{1}{\alpha}\left[1 + \xi^2 f(\kappa^2)\right]^{\frac{1}{2}} -i\nu/2$$

$$\chi_5 = -i\nu f(\kappa^2) \qquad ,$$

where again $f(\kappa^2) = \kappa^2/(\kappa^2 + 1)$. The residue of χ_5 provides the only contribution to the return current. The $\emptyset$ integration can be easily done, and after calculating the residue for χ_5 the expression for the return current becomes

$$j_z'(R,\theta,Z) = -A \int_0^\infty d\kappa \, \frac{J_1(\kappa A)\left[J_o(\kappa R) - \xi\kappa \sin\theta J_1(\kappa R)\right]}{(\kappa^2 + 1)\left[1 + \xi^2 f(\kappa^2)\right]} \cdot \exp\left[-\nu Z\kappa^2/(\kappa^2 + 1)\right] \quad (5.3)$$

If we put $\xi = 0$, Eq. (5.3) will reduce to the expression for the current with no magnetic field, Eq. (3.22). For finite ξ we see that the return current is no longer azimuthally symmetric but, to this order in the expansion, has a component which is proportional to $\sin\theta$. We show later that this component is nothing more than the perturbed charge density streaming by with the beam velocity.

Again we will assume there is a range of Z where the exponential is approximately equal to 1. Equation (5.3) then becomes

$$j_z'(R,\theta,Z) = -A \int_0^\infty d\kappa \, \frac{J_1(\kappa A)\left[J_o(\kappa R) - \xi\kappa \sin\theta J_1(\kappa R)\right]}{(\kappa^2 + 1)\left[1 + \xi^2 f(\kappa^2)\right]}$$

$$= \begin{cases} -1 + J_1(R) + J_2(R)\sin\theta\,, & R < A \\ + J_3(R) + J_4(R)\sin\theta\,, & R > A \end{cases} \quad , \quad (5.4)$$

where

$$J_1(R) = \frac{A}{(1 + \xi^2)^{1/2}} I_o\left[R/(1 + \xi^2)^{1/2}\right]K_1\left[A/(1 + \xi^2)^{1/2}\right]$$

$$J_2(R) = \frac{\xi A}{(1 + \xi^2)} I_1\left[R/(1 + \xi^2)^{1/2}\right]K_1\left[A/(1 + \xi^2)^{1/2}\right]$$

$$J_3(R) = \frac{A}{(1 + \xi^2)^{1/2}} I_1\left[A/(1 + \xi^2)^{1/2}\right]K_o\left[R/(1 + \xi^2)^{1/2}\right]$$

$$J_4(R) = \frac{\xi A}{(1 + \xi^2)} I_1\left[A/(1 + \xi^2)^{1/2}\right]K_1\left[R/(1 + \xi^2)^{1/2}\right] \quad .$$

The ratio of the net current within the beam to the beam current is

$$I_{net}/I_{beam} = \frac{1}{\pi A^2}\int_0^A RdR \int_0^{2\pi} d\phi \left[J_1(R) + \sin\theta\, J_2(R)\right]$$

$$= 2I_1\left[A/(1 + \xi^2)^{1/2}\right] K_1\left[A/(1 + \xi^2)^{1/2}\right] \quad . \qquad (5.5)$$

For $A^2 >> (1 + \xi^2)$, the new condition for current neutralization, we have

$$I_{net}/I_{beam} \approx \frac{(1 + \xi^2)^{1/2}}{A} \quad . \qquad (5.6)$$

Using the same arguments as before the current decay length will be the same as before, i.e. $L_n \sim A^2/\nu$.

From Eq. (2.18) we can also calculate the perturbed charge density. Following the same procedure used in calculating the current we get

$$\rho'(R,\theta,Z) = \begin{cases} -1 + \rho_1(R)\,, & R < A \\ \rho_2(R)\,, & R > A \end{cases}\,, \qquad (5.7)$$

where

$$\rho_1(R) = \frac{\xi A}{(1 + \xi^2)}\, I_1\left[R/(1 + \xi^2)^{\frac{1}{2}}\right] K_1\left[A/(1 + \xi^2)^{\frac{1}{2}}\right]$$

$$\rho_2(R) = \frac{\xi A}{(1 + \xi^2)}\, I_1\left[A/(1 + \xi^2)^{\frac{1}{2}}\right] K_1\left[R/(1 + \xi^2)^{\frac{1}{2}}\right] .$$

The above expression is valid for values of Z where the slow decay of the return current can be neglected. We now see that if there were no magnetic field, i.e. $\xi = 0$, then the net charge density would be zero. However, because of the magnetic field, a space charge is present. The sin θ term in Eq. (5.7) is identical to the sin θ term in Eq. (5.4), that is this component of the current is nothing more than the space charge streaming by with the beam velocity. The space charge electric field generated by the charge density provides the necessary mechanism to transport the return current across the magnetic field. The electric field generated has just the proper character to cancel out the magnetic force on the return current. If one makes the lengthy,but straight forward calculation of the Lorentz force on a return current electron (see Appendix E), one finds that after the plasma oscillations have decayed away the electron moves in a force free region and the return current will flow until it is diffused away by collisions.

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FOOTNOTES

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FIGURE CAPTIONS

Fig. 1 Schematic description of the z dependence of the net current density in the laboratory frame in region I the current oscillations are damped out on a scale length of order $v_o\tau$. In region II the beam is current neutralized. In region III collision have damped out the plasma current and the net current is now equal to the beam current.

Fig. 2 Schematic description of the values of the zeros in the complex plane . The arrows indicate the direction the zeros move as κ goes from 0 to ∞.

APPENDIX A

In this appendix the matrix elements of $\underset{\approx}{S}(s,\underset{\sim}{k})$ are given for injection of the beam parallel and perpendicular to the static magnetic field. Since we are interested in the time asymptotic solutions, we have set s=0 in all the matrix elements except where it is a multiplicative factor. The elements of $\underset{\approx}{S}(s,\underset{\sim}{k})$ are then

1. Parallel Injection

$$S_{11} = \frac{-s^2}{k_z^2 v_o^2 (1 + i\varepsilon)} \left[\frac{k_\perp^2 + \gamma_o^{-2} k_z^2}{k^2} - H(\Omega_\parallel^2) \frac{k_\perp^2}{k^2} \right]$$

$$S_{12} = \frac{-is}{k_z v_o (1 + i\varepsilon)} \frac{k_\perp}{k_z} \left[1 - H(\Omega_\parallel^2) \right]$$

$$S_{13} = \frac{-s}{\Omega} \frac{\gamma_o k_\perp}{k} H(\Omega_\parallel^2)$$

$$S_{21} = S_{12}(\Omega_\parallel \rightarrow -\Omega_\parallel)$$

$$S_{22} = \frac{1}{(1 + i\varepsilon)} \frac{k^2}{k_z^2} \left[1 - H(\Omega_\parallel^2) \right]$$

$$S_{23} = -i\gamma_o \frac{k v_o}{\Omega_\parallel} H(\Omega_\parallel^2)$$

$$S_{31} = S_{13}(\Omega_\parallel \rightarrow -\Omega_\parallel)$$

$$S_{32} = S_{23}(\Omega_\parallel \rightarrow -\Omega_\parallel)$$

$$S_{33} = \frac{1}{(1 + i\varepsilon)} \left[1 - H(\Omega_{\parallel}^{2})\right] \quad \text{(A.1)}$$

where $H(\Omega^2) = \Omega^2 / \left[\Omega^2 - \gamma_o^{\,2} k_z v_o^{\,2} (1 + i\varepsilon)^2\right]$, $\varepsilon = 1/k_z v_o \tau$,

and

2. Perpendicular Injection

$$S_{11} = \frac{-s^2}{k_z^{\,2} v_o^{\,2} (1 + i\varepsilon)} \left[\frac{k_\perp^{\,2} + \gamma_o^{\,-2} k_z^{\,2}}{k^2} - H(\Omega_\perp^{\,2}) \frac{k_y^{\,2} + \gamma_o^{\,-2} k_z^{\,2}}{k^2}\right]$$

$$S_{12} = \frac{-is}{k_z v_o (1 + i\varepsilon)} \left\{\frac{k_\perp}{k_z} - H(\Omega_\perp^{\,2}) \left[\frac{k_y^{\,2}}{k_z k_\perp} + i\frac{k_y}{k_\perp} \frac{k_z v_o (1+i\varepsilon)}{\Omega_\perp}\right]\right\}$$

$$S_{13} = \frac{is}{k_z v_o (1 + i\varepsilon)} H(\Omega_\perp^{\,2}) \left[\frac{k_x k_y}{k k_\perp} + i\frac{k_x k_z}{k k_\perp} \frac{k_x v_o (1 + i\varepsilon)}{\Omega_\perp}\right]$$

$$S_{21} = S_{12}(\Omega_\perp \rightarrow -\Omega_\perp)$$

$$S_{22} = \frac{1}{(1 + i\varepsilon)} \frac{k^2}{k_z^{\,2}} \left[1 - H(\Omega_\perp^{\,2}) \frac{k_y^{\,2}}{k_\perp^{\,2}}\right]$$

$$S_{23} = \frac{1}{(1 + i\varepsilon)} \frac{k_x k_y k}{k_z k_\perp^{\,2}} H(\Omega_\perp^{\,2})$$

$$S_{31} = S_{13}(\Omega_\perp \rightarrow -\Omega_\perp)$$

$$S_{32} = S_{23}(\Omega_\perp \rightarrow -\Omega_\perp)$$

$$S_{33} = \frac{1}{(1 + i\varepsilon)} \left[1 - H(\Omega_\perp^{\,2}) \frac{k_x^{\,2}}{k_\perp^{\,2}}\right] \quad . \quad \text{(A.2)}$$

APPENDIX B

In this appendix we give the elements necessary to represent $\underline{\underline{Y}}^{-1}(s,\underset{\sim}{k})$ as a cofactor matrix, $\underline{\underline{R}}(s,\underset{\sim}{k})$, divided by the determinant of $\underline{\underline{Y}}(s,\underset{\sim}{k})$. Because our source term, the electrostatic field from the unneutralized beam, has only a component in the $\hat{e}_1$ direction, only three elements of $\underline{\underline{R}}(s,\underset{\sim}{k})$ are needed and hence only these three will be given. The elements of $\underline{\underline{R}}(s,\underset{\sim}{k})$ are then

1. Parallel Injection

$$R_{11} = \frac{1}{(1+i\varepsilon)^2}\frac{k^2}{k_z^2}\left\{\left[k_z^2(1+i\varepsilon)+\omega_p^2/c^2\right]\right.$$

$$\left.\cdot\left[k^2(1+i\varepsilon)+\omega_p^2/c^2\right] - H(\Omega_{\parallel}^2)\frac{\omega_p^2}{c^2}\left[(k^2+k_z^2)(1+i\varepsilon)+\omega_p^2/c^2\right]\right\}$$

$$R_{21} = \frac{is}{k_z v_o(1+i\varepsilon)^2}\frac{k_\perp}{k_z}\frac{\omega_p^2}{c^2}\left[k^2(1+i\varepsilon)+\omega_p^2/c^2\right]\left[1-H(\Omega_{\parallel}^2)\right]$$

$$R_{31} = \frac{-s}{\Omega_{\parallel}}\frac{\gamma_o\omega_p^2}{c^2}kk_\perp \quad , \qquad \text{(B.1)}$$

and

$$\left|\underline{\underline{Y}}(s,\underset{\sim}{k})\right| = \frac{s^2}{k_z^2c^2(1+i\varepsilon)^3}\left\{\left[k^2(1+i\varepsilon)+\omega_p^2/c^2\right]^2\right.$$

$$\cdot\left[k_z^2(1+i\varepsilon)-\omega_p^2/\alpha^2c^2\right] - H(\Omega_{\parallel}^2)\left\{\left[k^2(1+i\varepsilon)+\omega_p^2/c^2\right]\right.$$

$$\cdot\left[k_z^2(1+i\varepsilon)-\omega_p^2/c^2\right] - k^2k_z^2(1+i\varepsilon)^2 \qquad \text{(B.2)}$$

$$\left.\left.\cdot\left[k^2(1+i\varepsilon)-\omega_p^2/c^2\right]\right\}\right\} \quad ,$$

and

2. Perpendicular Injection

$$R_{11} = \frac{1}{(1 + i\varepsilon)^2} \frac{k^2}{k_z^2} \left\{ \left[k_z^2 (1 + i\varepsilon) + \omega_p^2/c^2\right] \right.$$

$$\left. \cdot \left[k^2 (1 + i\varepsilon) + \omega_p^2/c^2\right] - H(\Omega_\perp^2) \frac{\omega_p^2}{c^2} \left[(k^2 - k_x^2)(1 + i\varepsilon) + \omega_p^2/c^2\right] \right\}$$

$$R_{21} = \frac{is}{k_z v_o (1 + i\varepsilon)^2} \frac{k_\perp}{k_z} \frac{\omega_p^2}{c^2} \left\{ \left[k^2 (1 + i\varepsilon) + \omega_p^2/c^2\right] \right.$$

$$- H(\Omega_\perp^2) \left\{ \frac{k_x^2}{k_\perp^2} \frac{\omega_p^2}{c^2} + \left[k^2 (1 + i\varepsilon) + \omega_p^2/c^2\right] \right.$$

$$\left. \left. \cdot \left[\frac{k_y^2}{k_\perp^2} - i\frac{k_z}{k_\perp^2} \frac{k_y v_o (1 + i\varepsilon)}{\Omega_\perp}\right] \right\} \right\}$$

$$R_{31} = \frac{-is}{k_z v_o (1 + i\varepsilon)} \frac{\omega_p^2}{c^2} \frac{k}{k_\perp} H(\Omega_\perp^2) \left\{ k_x k_y \right.$$

$$\left. -i \left[k_z^2 (1 + i\varepsilon) + \omega_p^2/c^2\right] \frac{k_x}{k_z} \frac{k_z v_o}{\Omega_\perp} \right\} , \qquad \text{(B.3)}$$

and

$$\left|\underline{\underline{Y}}(s,k)\right| = \frac{s^2}{k_z^2 c^2 (1 + i\varepsilon)^3} \left\{ \left[k^2 (1 + i\varepsilon) + \omega_p^2/c^2\right]^2 \right.$$

$$\cdot \left[k_z^2 (1 + i\varepsilon) - \omega_p^2/\alpha^2 c^2\right]$$

$$-H(\Omega_{\perp}^{2})\left\{\left[k^{2}(1+i\varepsilon)+\omega_{p}^{2}/c^{2}\right]^{2}\left[k_{z}^{2}(1+i\varepsilon)-\omega_{p}^{2}/\alpha^{2}c^{2}\right]\right.$$

$$-k^{2}k_{z}^{2}(1+i\varepsilon)^{2}\left[k^{2}(1+i\varepsilon)+\omega_{p}^{2}/c^{2}\right]$$

$$\left.+\frac{\omega_{p}^{2}}{\alpha^{2}c^{2}}k_{x}^{2}k^{2}(1+i\varepsilon)\right\}\Bigg\} \,. \qquad (B.4)$$

APPENDIX C

Only three elements of the matrix $\underline{\underline{Q}}(s,\underset{\sim}{k}) = \underline{\underline{S}}(s,\underset{\sim}{k}) \cdot \underline{\underline{R}}(s,\underset{\sim}{k})$ are needed, these are

1. Parallel Injection

$$Q_{11} = \frac{-s^2}{k_z^{\,2} v_o^{\,2}(1+i\varepsilon)^3}\Bigg\{\left[k_z^{\,2}(1+i\varepsilon)+\omega_p^{\,2}/c^2\right]$$

$$\cdot\left[k^2(1+i\varepsilon)+\omega_p^{\,2}/c^2\right]\frac{1}{k_z^{\,2}}\left[k_\perp^{\,2}+\gamma_o^{-2}k_z^{\,2}\right]$$

$$-\frac{k_\perp^{\,2}}{k_z^{\,2}}\frac{\omega_p^{\,2}}{c^2}\left[k^2(1+i\varepsilon)+\omega_p^{\,2}/c^2\right] - H(\Omega_\parallel^{\,2})$$

$$\cdot\Bigg\{k_\perp^{\,2}(1+i\varepsilon)\left[k_z^{\,2}(1+i\varepsilon)+\omega_p^{\,2}/c^2\right]+\frac{\omega_p^{\,2}}{\gamma_o^{\,2}c^2}$$

$$\cdot\left[(k^2+k_z^{\,2})(1+i\varepsilon)+\omega_p^{\,2}/c^2\right]\Bigg\}\Bigg\}$$

$$Q_{21} = \frac{-is}{k_z v_o(1+i\varepsilon)^3}\frac{k_\perp}{k_z}k^2(1+i\varepsilon)\left[k^2(1+i\varepsilon)+\omega_p^{\,2}/c^2\right]$$

$$\cdot\left[1-H(\Omega_\parallel^{\,2})\right]$$

$$Q_{31} = 0 \quad , \qquad \text{(C.1)}$$

and

2. Perpendicular Injection

$$Q_{11} = \frac{-s^2}{k_z^2 v_o^2 (1 + i\varepsilon)^3} \Bigg\{ \left[k_z^2 (1 + i\varepsilon) + \omega_p^2/c^2\right]$$

$$\cdot \left[k^2 (1 + i\varepsilon) + \omega_p^2/c^2\right] \frac{1}{k_z^2} \left[k^2 + \gamma_o^{-2} k_z^2\right]$$

$$- \frac{k_\perp^2}{k_z^2} \frac{\omega_p^2}{c^2} \left[k^2 (1 + i\varepsilon) + \omega_p^2/c^2\right] - H(\Omega_\perp^2)$$

$$\cdot \Bigg\{ \left[k^2 (1 + i\varepsilon) + \omega_p^2/c^2\right] \left[k_z^2 (1 + i\varepsilon) + \omega_p^2/c^2\right]$$

$$\cdot \frac{1}{k_z^2} \left[k_y^2 + \gamma_o^{-2} k_z^2\right] - \frac{k_\perp^2}{k_z^2} \frac{\omega_p^2}{c^2} \left[(k^2 - k_x^2)(1 + i\varepsilon) + \omega_p^2/c^2\right] \Bigg\} \Bigg\}$$

$$Q_{21} = \frac{-is}{k_z v_o (1 + i\varepsilon)^3} \frac{k_\perp}{k_z} k^2 (1 + i\varepsilon) \Bigg\{ \left[k^2 (1 + i\varepsilon) + \omega_p^2/c^2\right]$$

$$-H(\Omega_\perp^2) \Bigg\{ \frac{k_y^2}{k_z^2} k^2 (1 + i\varepsilon) \left[k^2 (1 + i\varepsilon) + \omega_p^2/c^2\right]$$

$$+ \frac{k_x^2}{k_z^2} \frac{\omega_p^2}{c^2} k^2 (1 + i\varepsilon) - ik^2 (1 + i\varepsilon) \left[k^2 (1 + i\varepsilon) + \omega_p^2/c^2\right]$$

$$\cdot \frac{k_y v_o (1 + i\varepsilon)}{\Omega_\perp} \Bigg\} \Bigg\}$$

$$Q_{31} = \frac{is}{k_z v_o (1 + i\varepsilon)} \frac{k_x k^3}{k_\perp} H(\Omega_\perp^2) \left\{ k_y - \frac{i v_o}{\Omega_\perp} \left[k_z^2 (1 + i\varepsilon) + \omega_p^2/c^2\right] \right\}$$

(C.2)

APPENDIX D

In this appendix we give a brief discussion of the behavior of the zeros of the polynomial in the denominator of the integrand in Eq. (4.1) for case (b), the large magnetic field ordering. In this ordering we have $\zeta \sim \varepsilon^{-1}$, $\alpha \sim 1$, and $\nu \sim \varepsilon^2$. The polynomial is

$$\left[(\chi^2 + \kappa^2)(\chi + i\nu) + \chi\right]^2 \left[\chi(\chi + i\nu) - 1/\alpha^2\right]$$

$$- \zeta^2\alpha^{-2}\chi(\chi^2 + \kappa^2) \left[(\chi^2 + \kappa^2)(\chi + i\nu) - \chi/\alpha^2\right] . \quad \text{(D.1)}$$

Figure (2) gives a schematic description of the values of the zeros in the complex plane. The arrows indicate the direction the zeros move as κ goes from 0 to ∞.

There are six zeros in the LHP and two zeros in the UHP. χ_1 and χ_2 begin at the origin and move down the negative imaginary axis as κ increases. χ_1 quickly reaches its asymptotic value of $\chi_1 \approx i\nu/\zeta^2$ and χ_2 moves monotonically down the imaginary axis to $-i\infty$ as κ goes to ∞. χ_3 and χ_4 are a conjugate pair of zeros which begin at $\pm 1/\alpha - i\nu/2$. As κ increases they move in towards the imaginary axis and become pur imaginary for $\kappa \approx 1/\alpha$. One pole then moves down the imaginary axis to $-i\infty$ and the other moves up the axis reaching its asymptotic limit, $-i\nu$, as κ goes to ∞. χ_5 and χ_6 are approximately located at

$\pm\ \zeta/\alpha\ - i\nu$ and their value does not change significantly for all values of κ. χ_7 and χ_8 are a conjugate pair of zeros in the UHP which begin at $\pm\ \alpha/\zeta + i\nu\alpha^2/\zeta^2$. As κ increases they move towards the positive imaginary axis and become pure imaginary for $\kappa \approx \alpha/\zeta$. They then both move up the imaginary axis, one more slowly at first than the other, towards $+i\infty$ as κ goes to ∞.

The zeros of (D.1) have been calculated using expansions in the parameters ζ^{-1} and ν for all values of κ. These expressions are not uniformly valid for all κ since the character of the conjugate pair zeros change dramatically when they come near the imaginary axis. At these two values of κ, $\kappa \approx \alpha/\zeta$ and $\kappa \approx 1/\alpha$, the expansions fail. One can reorder the terms in the polynomial at these singular points and calculate the values of the zeros. One then obtains a complete picture of the behavior of the zeros. This has been done but we will not give any expressions here since no insight is gained from this presentation.

APPENDIX E

In this appendix we will calculate the $\hat{a}_y$ component of the Lorentz force on the plasma electrons in the laboratory frame, i.e. $E_y' + v_z' B_{xo}/c$, in order to show that the space charge electric field does indeed cancel the magnetic force on the return current electrons. First consider the perturbed velocity, v_z'. The solution to Eq. (2.6) may be written as follows,

$$\underset{\sim}{v}(s,\underset{\sim}{k}) = -\frac{e}{\gamma_o m_o s} \underset{\approx}{C}(s,\underset{\sim}{k}) \cdot \underset{\sim}{E}(s,\underset{\sim}{k}) \quad , \tag{E.1}$$

where the elements of $\underset{\approx}{C}(s,\underset{\sim}{k})$ are

$$C_{11} = \frac{is}{k_z v_o(1+i\varepsilon)} \left\{ \frac{k_\perp^2 + \gamma_o^{-2} k_z^2}{k^2} - H(\Omega_\perp^2) \left[\frac{k_y^2 + \gamma_o^{-2} k_z^2}{k^2} \right] \right\}$$

$$C_{12} = -\frac{1}{(1+i\varepsilon)} \frac{k_\perp}{k_z} + H(\Omega_\perp^2) \left[\frac{1}{(1+i\varepsilon)} \frac{k_y^2}{k_z k_\perp} + i \frac{k_z v_o}{\Omega_\perp} \frac{k_y}{k_\perp} \right]$$

$$C_{13} = H(\Omega_\perp^2) \left[\frac{1}{(1+i\varepsilon)} \frac{k_x k_y}{k k_\perp} + i \frac{k_z v_o}{\Omega_\perp} \frac{k_x k_z}{k k_\perp} \right]$$

$$C_{21} = \frac{-is}{k_z v_o(1+i\varepsilon)} \left\{ \beta_o^2 \frac{k_z k_\perp}{k^2} - H(\Omega_\perp^2) \left[\frac{k_y^2 k_z}{k^2 k_\perp} - \frac{1}{\gamma_o^2} \frac{k_z k_\perp}{k^2} - i \frac{k_z v_o}{\Omega_\perp} \cdot \frac{k_y(1+i\varepsilon)}{k_\perp} \right] \right\}$$

$$C_{22} = \frac{1}{(1+i\varepsilon)} - H(\Omega_\perp^2) \left[\frac{1}{(1+i\varepsilon)} \frac{k_y^2}{k_\perp^2} - i \frac{k_y v_o}{\Omega_\perp} \right]$$

$$C_{23} = -H(\Omega_\perp^2) \left[\frac{1}{(1+i\varepsilon)} \frac{k_x k_y k_z}{k^2 k_\perp} - i \frac{k_z v_o}{\Omega_\perp} \frac{k_x}{k} \right]$$

$$C_{31} = \frac{is}{k_z v_o(1 + i\varepsilon)} H(\Omega_\perp^2) \left[\frac{k_x k_y}{kk_\perp} - i \frac{k_z v_o}{\Omega_\perp} \frac{k_x k_z (1 + i\varepsilon)}{kk_\perp}\right]$$

$$C_{32} = - H(\Omega_\perp^2) \left[\frac{1}{(1 + i\varepsilon)} \frac{kk_x k_y}{k_z k_\perp^2}\right]$$

$$C_{33} = \frac{1}{(1 + i\varepsilon)} - H(\Omega_\perp^2) \left[\frac{1}{(1 + i\varepsilon)} \frac{k_y^2}{k_\perp^2}\right] .$$

The perturbed velocity in the $\hat{a}_z$ direction is given by

$$v_z(s,\underset{\sim}{k}) = \frac{k_z}{k} v_1(s,\underset{\sim}{k}) + \frac{k_\perp}{k} v_2(s,\underset{\sim}{k}) . \tag{E.2}$$

After substituting Eq. (E.1) into Eq. (E.2) the expression for the perturbed velocity becomes

$$v_z(t\to\infty,\underset{\sim}{k}) = \lim_{s\to 0} \frac{-e}{\gamma_o m_o} \left\{\frac{is}{k_z v_o(1 + i\varepsilon)} \frac{k_z}{k}\right.$$

$$\cdot \left\{\gamma_o^{-2} - H(\Omega_\perp^2) \left[\gamma_o^{-2} + i \frac{k_z v_o}{\Omega_\perp} \frac{k_y(1 + i\varepsilon)}{k}\right]\right\} E_1(s,\underset{\sim}{k}) \tag{E.3}$$

$$\left. + i H(\Omega_\perp^2) \frac{k_y v_o}{\Omega_\perp} \frac{k}{k_\perp} E_2(s,\underset{\sim}{k}) + i H(\Omega_\perp^2) \frac{k_x v_o}{\Omega_\perp} \frac{k_z}{k_\perp} E_3(s,\underset{\sim}{k})\right\} .$$

Using Eq. (2.10) to evaluate $E_1(s,\underset{\sim}{k})$, $E_2(s,\underset{\sim}{k})$ and $E_3(s,\underset{\sim}{k})$ in Eq. (E.3) we have

$$v_z(t\to\infty,\underset{\sim}{k}) = \frac{4\pi e}{\gamma_o m_o} \frac{s^2}{\Omega_\perp^2} \frac{\beta_o \rho_b(k)}{|\underline{\underline{Y}}(s,\underset{\sim}{k})|} \frac{\left[k^2(1 + i\varepsilon) + \omega_p^2/c^2\right]}{(1 + i\varepsilon)}$$

$$\cdot H(\Omega_\perp^2) \left[k_z^2(1 + i\varepsilon) + \omega_p^2/c^2 + i \frac{k_y \Omega_\perp}{v_o}\right] . \tag{E.4}$$

The above expression is for the perturbed velocity in the beam frame. In the laboratory frame we have

$$v_z'(s,\underset{\sim}{k}) = \gamma_o^{\,2} v_z(s,\underset{\sim}{k}) \quad . \tag{E.5}$$

The magnetic force then becomes

$$v_z'(t\to\infty,\underset{\sim}{k})B_{xo}/c = \frac{4\pi s^2}{c}\,\frac{\alpha}{\Omega_\perp}\,\frac{\rho_b(k)}{|\underline{\underline{Y}}(s,k)|}\,\frac{\left[k^2(1+i\varepsilon)+\omega_p^{\,2}/c^2\right]}{(1+i\varepsilon)}$$

$$\cdot\, H(\Omega_\perp^{\,2})\left[k_z^{\,2}(1+i\varepsilon)+\omega_p^{\,2}/c^2+i\,\frac{k_y\Omega}{v_o}\right] \quad . \tag{E.6}$$

In the laboratory frame the $\hat{a}_y$ component of the electric field is given by

$$E_y'(s,\underset{\sim}{k}) = \gamma_o\left[E_y(s,\underset{\sim}{k}) - \beta_o B_x(s,\underset{\sim}{k})\right] \quad . \tag{E.7}$$

The $\hat{a}_y$ component of the electric field in the beam frame is given by

$$E_y(t\to\infty,\underset{\sim}{k}) = \lim_{s\to 0}\frac{sk_y}{k}E_1(s,\underset{\sim}{k}) \quad , \tag{E.8}$$

and the $\hat{a}_x$ component of the magnetic field by

$$B_x(t\to\infty,\underset{\sim}{k}) = \lim_{s\to 0} s\left[-\frac{k_x k_z}{kk_\perp}B_2(s,\underset{\sim}{k}) + \frac{k_y}{k_\perp}B_3(s,\underset{\sim}{k})\right] \quad . \tag{E.9}$$

Using Eq. (2.1) to evaluate $B_2(s,\underset{\sim}{k})$ and $B_3(s,\underset{\sim}{k})$ in terms of $E_2(s,\underset{\sim}{k})$ and $E_3(s,\underset{\sim}{k})$ we have

$$B_x(t\to\infty,\underset{\sim}{k}) = \lim_{s\to 0} -ic\left[\frac{k_y k}{k_\perp} E_2(s,\underset{\sim}{k}) + \frac{k_x k_z}{k_\perp} E_3(s,\underset{\sim}{k})\right] \tag{E.10}$$

Substituting Eqs. (E.8) and (E.10) into Eq. (E.7), and using Eq. (2.10) to evaluate the electric field, Eq. (E.7) becomes

$$E_y'(t\to\infty,k) =$$

$$-\frac{4\pi s^2}{c}\frac{\alpha}{\Omega_\perp}\frac{\rho_b(\underset{\sim}{k})}{|\underline{\underline{Y}}(s,\underset{\sim}{k})|}\frac{\left[k^2(1+i\varepsilon)+\omega_p{}^2/c^2\right]}{(1+i\varepsilon)} H(\Omega_\perp{}^2)$$

$$\cdot\left[i\,\frac{k_y\Omega_\perp}{v_o} + \omega_p{}^2/c^2 - i\,\gamma_o{}^2\,\frac{k_y v_o}{\Omega_\perp}\,k_z{}^2(1+i\varepsilon)^2\right] \tag{E.11}$$

Adding Eqs.(E.6) and (E.11), the expression for the Lorentz force becomes

$$E_y'(\underset{\sim}{k}) + v_z'(\underset{\sim}{k})B_{xo}/c = \frac{4\pi s^2}{c}\frac{\alpha}{\Omega_\perp}\frac{\rho_b(k)}{|\underline{\underline{Y}}(s,k)|}\frac{\left[k^2(1+i\varepsilon)+\omega_p{}^2/c^2\right]}{(1+i\varepsilon)}$$

$$\cdot\, H(\Omega_\perp{}^2)\left[i\,\frac{k_y v_o}{\Omega_\perp}\,\gamma_o{}^2 k_z{}^2(1+i\varepsilon)^2 + k_z{}^2(1+i\varepsilon)\right] \quad . \tag{E.12}$$

Taking the inverse Fourier transform and using Eqs. (2.15) and (B.4) to evaluate $\rho_b(\underset{\sim}{k})$ and $|\underline{\underline{Y}}(s,\underset{\sim}{k})|$ we get

$$E_y'(\underset{\sim}{X}) + v_z'(\underset{\sim}{X})B_{xo}/c = -4\pi e\gamma_o N_b A \int_0^\infty \frac{d\kappa}{2\pi} J_1(\kappa A)$$

$$\cdot \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\left[i\kappa R \cos(\phi - \theta)\right] \int_{-\infty}^{0} d\tilde{Z} \int_{-\infty}^{+\infty} d\chi \exp\left[-i(\tilde{Z} + Z)\chi\right]$$

$$\cdot \left\{ \chi \left[(\chi^2+\kappa^2)(\chi+i\nu) + \chi\right]\left[i\kappa(\chi+i\nu)^2 \sin\phi + \frac{\beta_o}{\alpha^2}\chi(\chi+i\nu)\right]\right\}$$

$$\cdot \left\{\left[(\chi^2+\kappa^2)(\chi+i\nu) + \chi\right]^2\left[\chi(\chi+i\nu) - 1/\alpha^2\right] - \frac{\xi^2}{\alpha^2}\chi(\chi^2+\kappa^2)\right.$$

$$\left. \cdot \left[(\chi^2+\kappa^2)(\chi+i\nu) + \chi\right] + \frac{\xi^2}{\alpha^4}\kappa^2(\chi^2+\kappa^2)\cos^2\phi\right\}^{-1} . \qquad \text{(E.13)}$$

As before, all variables have been normalized to $\lambda_E = c/\omega_p$. Neglecting the $0(\alpha^{-4})$ term in the denominator of the integrand and the $0(\alpha^{-2})$ term in the numerator of the integrand, Eq. (E.13) reduces to

$$E_y'(\underset{\sim}{X}) + v_z'(\underset{\sim}{X})B_{ox}/c = -4\pi e\gamma_o N_b A \int_0^\infty \frac{d\kappa}{2\pi} J_1(\kappa A) \int_0^{2\pi} \frac{d\phi}{2\pi} \sin\phi$$

$$\cdot \exp\left[i\kappa R \cos(\phi - \theta)\right] \int_{-\infty}^{0} d\tilde{Z} \int_{-\infty}^{\infty} d\chi \exp\left[-i(\tilde{Z} + Z)\chi\right]$$

$$\cdot \frac{i\chi\kappa(\chi+i\nu)^2}{\left[(\chi^2+\kappa^2)(\chi+i\nu) + \chi\right]\left[\chi(\chi+i\nu) - 1/\alpha^2\right] - \frac{\xi^2}{\alpha^2}\chi(\chi^2+\kappa^2)} . \qquad \text{(E.14)}$$

Doing the χ and $\tilde{Z}$ integrations as before, we have

$$E_y'(\underset{\sim}{X}) + v_z'(\underset{\sim}{X})B_{ox}/c = -4\pi e\gamma_o N_b A\alpha^2\nu^2 \sin\theta$$

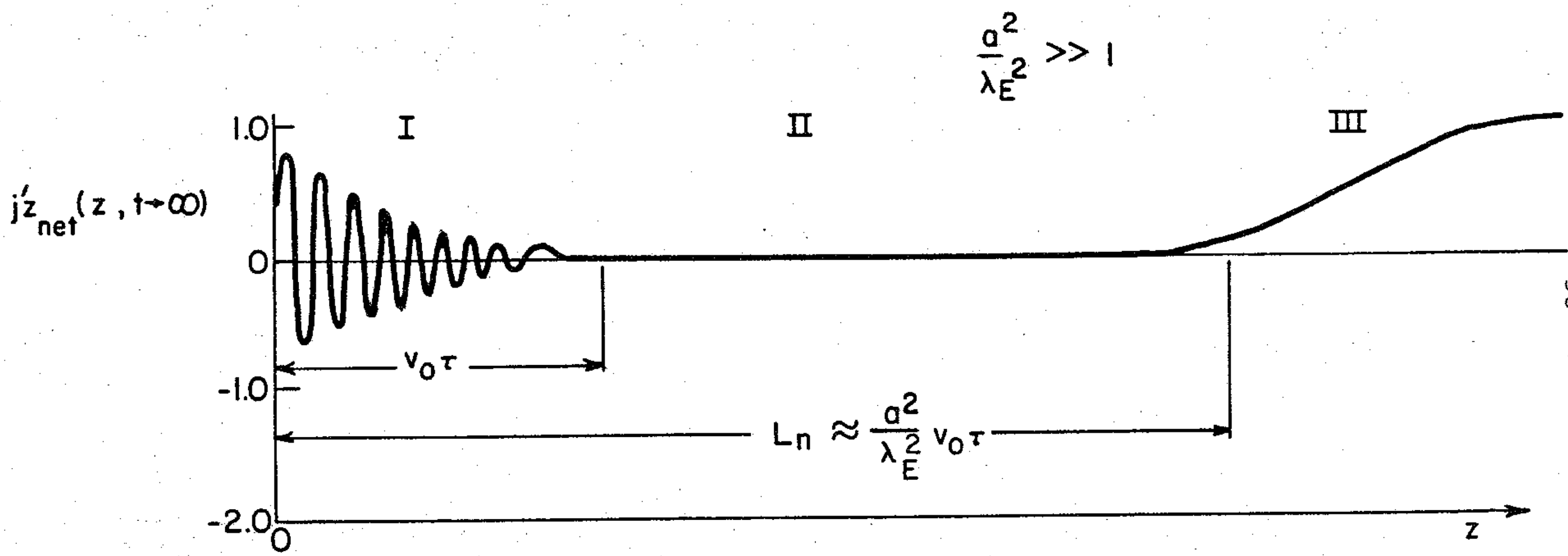
$$\cdot \int_0^\infty \frac{\kappa d\kappa \; J_1(\kappa A) \; J_1(\kappa R) \; \exp\left[-\nu Z f(\kappa^2)\right]}{(\kappa^2+1)^3\left[1 + \xi^2 f(\kappa^2)\right]} \quad , \qquad (E.15)$$

where the above is valid for $Z > 1/\nu$. We now see that in this region the Lorentz force term is order $\alpha^2\nu^2$. Since the momentum relaxation time in the laboratory frame is $\tau' = \tau/\gamma_o$, then $\alpha\nu = 1/\omega_p\tau' = \nu'\,\text{coll.}/\omega_p$, and the force on the return current electrons is negligible.

TABLE I

Summary of Neutralization Criteria

Magnetic Field	Neutralization Criterion	I_{net}/I_{beam}	L_n
$B = 0$	$a^2/\lambda_E^2 >> 1$	λ_E/a	$v_0 \tau a^2/\lambda_E^2$
$B = B_{\parallel}$	$a^2/\lambda_E^2(1+\zeta^2) >> 1$	$\frac{\left[(1+i\zeta)^{1/2}+(1-i\zeta)^{1/2}\right]}{2a}\lambda_E$	$v_0 \tau a^2/\lambda_E^2$
$B = B_{\perp}$	$a^2/\lambda_E^2(1+\xi^2) >> 1$	$\frac{(1+\xi^2)^{1/2}\lambda_E}{a}$	$v_0 \tau a^2/\lambda_E^2$



$\frac{a^2}{\lambda_E^2} >> 1$
I
II
III
1.0
0
-1.0
-2.0
0
$j'_{z_{net}}(z, t \to \infty)$
$v_0 \tau$
$L_n \approx \frac{a^2}{\lambda_E^2} v_0 \tau$
z

Fig. 1

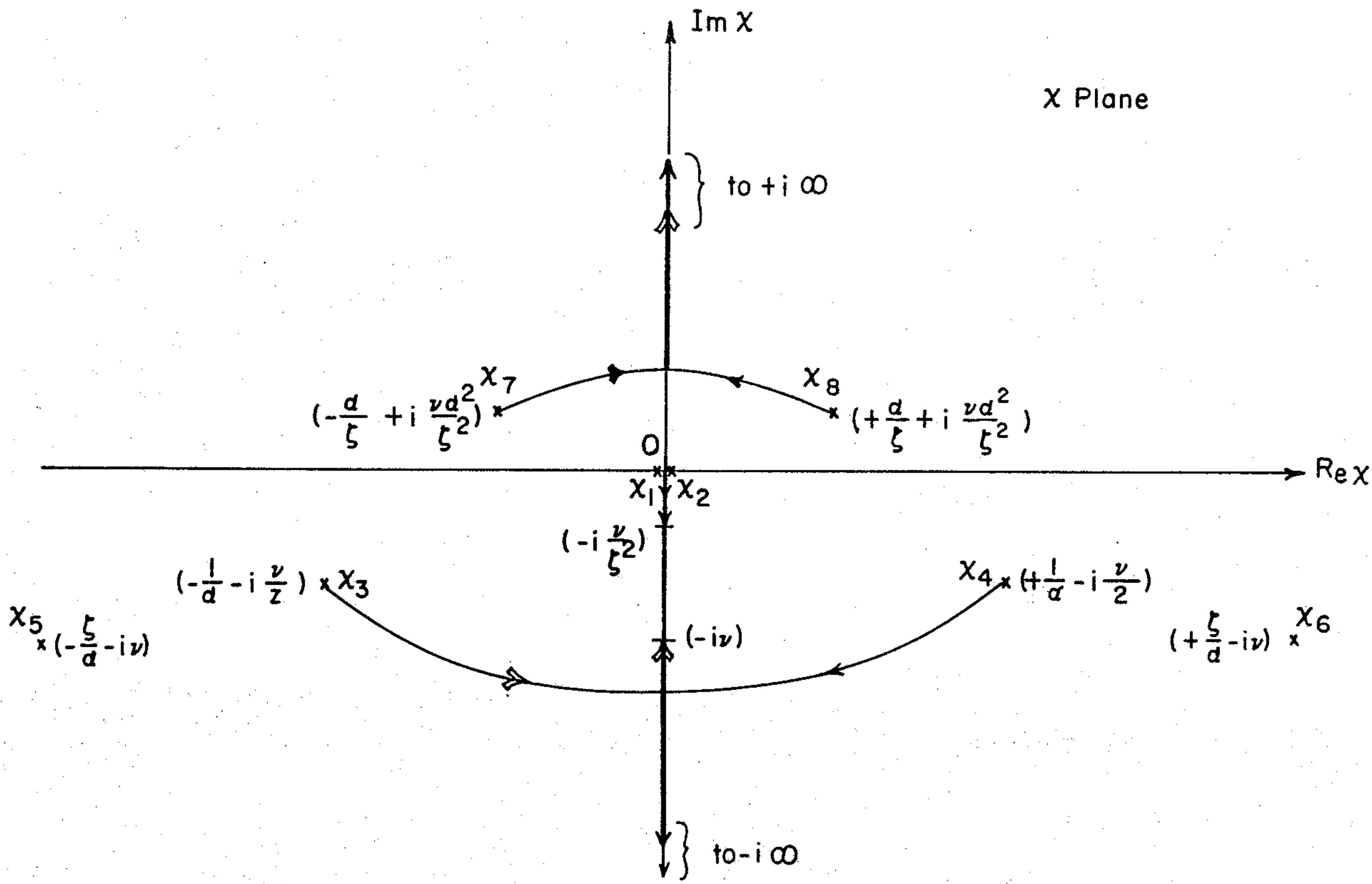

Im X
X Plane
to $+i\infty$
X_7 $(-\frac{\alpha}{\zeta} + i\frac{\nu\alpha^2}{\zeta^2})$
X_8 $(+\frac{\alpha}{\zeta} + i\frac{\nu\alpha^2}{\zeta^2})$
O
Re X
X_1
X_2
$(-i\frac{\nu}{\zeta^2})$
$(-\frac{1}{\alpha} - i\frac{\nu}{2})$ X_3
X_4 $(+\frac{1}{\alpha} - i\frac{\nu}{2})$
X_5 $(-\frac{\zeta}{\alpha} - i\nu)$
$(-i\nu)$
$(+\frac{\zeta}{\alpha} - i\nu)$ X_6
to $-i\infty$

Fig. 2